TANJONG KATONG GIRLS' SCHOOL
MID-YEAR EXAMINATION 2016
SECONDARY THREE

4047 **ADDITIONAL MATHEMATICS**

Wednesday **04 May 2016** **2 h 15 min**

Additional Materials: Answer Paper

READ THESE INSTRUCTIONS FIRST

Write your name, class and register number on all the work you hand in.

Write in dark blue or black pen on both sides of the paper, and use a pencil for drawing graphs and diagrams. Do not use staples, highlighters or correction fluid.

Answer **all** the questions.

Write your answers on the separate writing paper provided.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.

The use of a scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total marks for this paper is 90.

Setter : Mrs M Loy
Markers : Mr Koh MH, Miss Yeo LS, Mrs Loy, Mr Ang WJ

This Question Paper consists of 5 printed pages, including this page.

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1. Given that $Ax^3 + x^2 - 13x - 2 = (x+3)(x+B)(2x-1) + C$ for all values of x, find A, B and C. [4]

2. Given that $y = e^{\ln\sqrt{3}}$, show that $y = \sqrt{3}$.
Hence, without using a calculator, evaluate
$$\frac{e^{\ln\sqrt{3}} \times \frac{1}{2}\log_3 9}{\log_9 3}$$ [4]

3. Find the range of values of c in the exact form, for which $y = 2x + c$ meets the curve $y^2 - 2x^2 = -5$. Hence deduce the range of values of c for which there is no intersection point between the line and the curve. [5]

4. Solve $\log_{100}(1+x) = \lg 4x - \lg\sqrt{8}$. [6]

5. A chicken farm with a population of 1000 chickens was hard hit by bird flu in 2015. The spread of the bird flu is given by $S = \dfrac{1000}{1+e^{2-t}}$, where S is the number of chickens infected after t days.

(i) Deduce the number of chickens infected with the bird flu in the long run. [1]

(ii) Estimate the initial number of chickens infected with the bird flu, leaving your answer correct to the nearest integer. [2]

(iii) The chickens will be culled when at least 70% of the chickens are infected. Determine when culling will take place. [3]

6. A line and a curve are represented by $27^{2x} = \frac{1}{9}(3)^y$ and $(9^x)^y = 3$ respectively.
Given that the line intersects the curve at point A and point B, find the distance between the two points, A and B. [7]

7. **(a)** A toy car moved at a speed of $\left(2+\sqrt{3}\right)$ cm per second from point M to point N. Given that the distance covered was $\left(2\sqrt{75}-1\right)$ cm, find the time taken to move from point M to point N in the form $a\sqrt{3}+b$, where a and b are constants. [4]

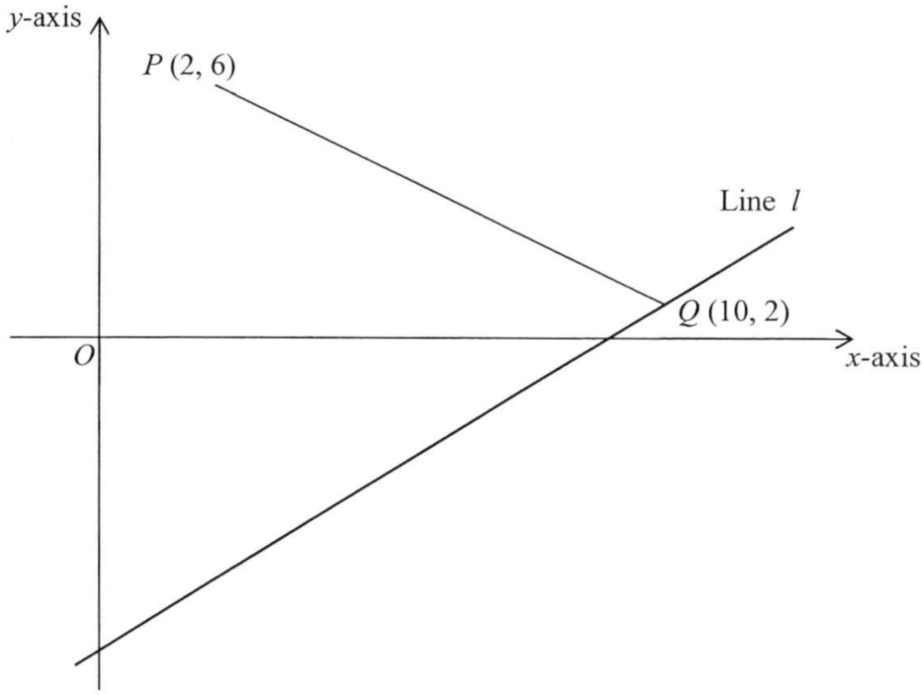
(b) Find the range of values of x that will satisfy the following inequalities,
$2x+5>4$ and $6-2x^2 \geq 3+x$. [4]

8. The quadratic equation $2x^2-2x-1=0$ has roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$.

(i) Find the value of $\alpha^2+\beta^2$. [4]

(ii) Find the quadratic equation in x whose roots are α^3 and β^3. [4]

9.



In the diagram, PQ is a straight line joining points $P(2\ ,\ 6)$ and $Q(10\ ,\ 2)$.
Line l is parallel to the line $2y=x-4$ and passes through point Q.
Given that the perpendicular bisector of PQ intersects Line l at point R,

(i) find the coordinates of point R , [6]

(ii) calculate the area of the quadrilateral $PQRO$. [3]

10. The equation of a curve $y = ax^2 + 2x + 6$ can be written in the form $y = 3(x + b)^2 + c$, where a, b and c are constants.

(i) State the value of a .

Expressing $y = ax^2 + 2x + 6$ in the form $y = 3(x + b)^2 + c$, show that $b = \frac{1}{3}$ and find c . [5]

Hence,

(ii) find the greatest value of $\frac{2}{y}$. Explain your choice for the value of y, [3]

(iii) determine with explanation the number of points of intersection between the curve and the x-axis. [2]

11. The polynomial $g(x) = x^3 + ax^2 - bx - 2$ has a factor $(x + 1)$ and it leaves a remainder of 24 when divided by $(x - 2)$.

(i) Show that $a = 4$ and $b = -1$. [4]

(ii) Taking $a = 4$ and $b = -1$, solve the equation $g(x) = 0$, leaving your answers in the exact form.
Hence, find the integer value of x for which $(x - 2)^3 + 4(x - 2)^2 + x - 4 = 0$. [7]

12. **(a)** Solve $5^{x+1} - 2(5^{-x}) = 9$. [6]

(b) Express $\frac{4x^3 + 2x - 1}{(2x - 1)(x + 1)^2}$ in partial fractions. [6]

End of Paper

Suggested Answer Key

1	$A=2$, $B=-2$, $C=-8$	**2**	$2\sqrt{3}$
3	$c \le -\sqrt{5}, c \ge \sqrt{5}$ $-\sqrt{5} < c < \sqrt{5}$	**4**	$x=1$ $x=-0.5$ (rejected)
5i	1 000	**5ii**	119 (nearest integer)
5iii	2.85 days		
6	$\frac{2\sqrt{37}}{3}$		
7a	Time taken = $21\sqrt{3}-32$	**7b**	$-\frac{1}{2} < x \le 1$
8i	$\alpha^2+\beta^2=8$	**8ii**	$x^2+20x-8=0$
9i	$R(\frac{10}{3},-\frac{4}{3})$	**9ii**	Area = 38 units^2
10i	$a=3$, $c=\frac{17}{3}$	**10ii**	For greatest value of $\frac{2}{y}$, y must be min value, that is $y=c$. $\therefore \frac{2}{y}=\frac{2}{\left(\frac{17}{3}\right)}=\frac{6}{17}$
10iii	Discriminant < 0 thus there is not real roots, so there is no intersection point.		
11i	Show question	**11ii**	$x=-1,\ x=\frac{-3\pm\sqrt{17}}{2}$ Integer value of $x=1$
12a	$x=0.431$	**12b**	$2+\frac{2}{9(2x-1)}-\frac{28}{9(x+1)}+\frac{7}{3(x+1)^2}$

1. Given that $Ax^3 + x^2 - 13x - 2 = (x+3)(x+B)(2x-1) + C$ for all values of x, find A, B and C. [4]

Comparing coefficient of x^3, $A = 2$ $2x^3 + x^2 - 13x - 2 = (x+3)(x+B)(2x-1) + C$ Put $x =\ \ 3$, $2(\ \ 3)^3 + (\ \ 3)^2 + 39\ \ \ 2 = C$ $C =\ \ 8$ Put $x = 0$, $3B + C =\ \ 2$ $3B\ \ \ 8 =\ \ 2$ $B =\ \ 2$	Choose appropriate value of x / expand and compare coefficients

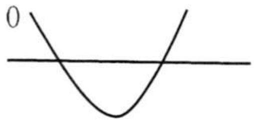
2. Given that $y = e^{\ln\sqrt{3}}$, show that $y = \sqrt{3}$.
Hence, without using a calculator, evaluate

$$\frac{e^{\ln\sqrt{3}} \times \frac{1}{2}\log_3 9}{\log_9 3}$$

[4]

$y = e^{\ln\sqrt{3}}$ $\ln y = \ln\sqrt{3}\ln e \quad (\ln e = 1)$ $\ln y = \ln\sqrt{3}$ $\therefore y = \sqrt{3}$ (shown)	Bring ln to both sides & obtain equation
$\dfrac{e^{\ln\sqrt{3}} \times \frac{1}{2}\log_3 9}{\log_9 3}$	Put $e^{\ln\sqrt{3}} = \sqrt{3}$
$= \dfrac{\sqrt{3} \times \frac{1}{2}\log_3 3^2}{\frac{1}{2}\log_9 9}$	Apply Log Law correctly -power law -$\log_3 3 = 1$
$= \sqrt{3} \times 2$ $= 2\sqrt{3}$	Answer in exact form

3. Find the range of values of c in the exact form, for which $y = 2x + c$ meets the curve $y^2 - 2x^2 = -5$. Hence deduce the range of values of c for which there is no intersection point between the line and the curve. [5]

Put $y = 2x + c$ into $y^2 - 2x^2 = -5$ $(2x+c)^2 - 2x^2 + 5 = 0$ $4x^2 + 4cx + c^2 - 2x^2 + 5 = 0$ $2x^2 + 4cx + c^2 + 5 = 0$	Combine to form a quadratic equation
For the line to meet the curve, discriminant ≥ 0 $(4c)^2 - 4(2)(c^2+5) \geq 0$ $16c^2 - 8c^2 - 40 \geq 0$ $8c^2 - 40 \geq 0$ 	Discriminant ≥ 0 *Deduct 1m if given discriminant >0*
$c^2 - 5 \geq 0$ $(c-\sqrt{5})(c+\sqrt{5}) \geq 0$ $c \leq -\sqrt{5},\ c \geq \sqrt{5}$ For no intersection points, $-\sqrt{5} < c < \sqrt{5}$	Obtain the factors

4. Solve $\log_{100}(1+x) = \lg 4x - \lg\sqrt{8}$. [6]

$\log_{100}(1+x) = \lg 4x - \lg\sqrt{8}$	Change of base to lg Apply Log Law to get $\lg\frac{4x}{\sqrt{8}}$ Apply Log Law to get lg $\left(\frac{16x^2}{8}\right)$

$\frac{\lg(1+x)}{\lg 10^2} = \lg\left(\frac{4x}{\sqrt{8}}\right)$ $\lg(1+x) = 2\lg\left(\frac{4x}{\sqrt{8}}\right)$ $\lg(1+x) = \lg\left(\frac{16x^2}{8}\right)$ $1+x = 2x^2$ $2x^2 = x+1$ $2x^2 - x - 1 = 0$ $(2x+1)(x-1) = 0$ $x = 1$, $x = -\frac{1}{2}$ (rejected) □ $\lg 4(-\frac{1}{2})$ is not defined.	Obtain factors If did not reject negative answer, no A1

5. A chicken farm with a population of 1000 chickens was hard hit by bird flu in 2015. The spread of the bird flu is given by $S = \frac{1000}{1+e^{2-t}}$, where S is the number of chickens infected after t days.

(i) Deduce the number of chickens infected with bird flu in the long run. [1]

As t becomes very large, e^{2-t} tends to 0. Number of chickens infected with flu = 1000	

(ii) Estimate the initial number of chickens infected with the bird flu, leaving your answer correct to the nearest integer. [2]

Put $t = 0$ $S = \frac{1000}{1+e^2}$ $S = 119.203$ Initial number of chickens infected with the bird flu = 119 (nearest integer)	Substitute $t = 0$

(iii) The chickens will be culled when at least 70% of the chickens are infected. Determine when culling will take place. [3]

$\dfrac{1000}{1+e^{2-t}} \geq 700$ $\dfrac{10}{7} \geq 1+e^{2-t}$ $2-t \leq -0.8473$ $t \geq 2.85$ After 2.85 days, culling will take place.	Form the linear inequality Solve linear inequality

6. A line and a curve are represented by $27^{2x} = \frac{1}{9}(3)^{y}$ and $(9^{x})^{y} = 3$ respectively. Given that the line intersects the curve at point A and point B, find the distance between the two points, A and B. [7]

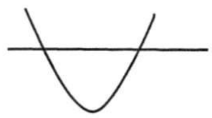
$27^{2x} = \frac{1}{9}(3)^{y}$ ———— (3) $3^{6x} = 3^{-2}(3^{y})$ ———— (2) $6x = y-2$ ———— (1) $(9^{x})^{y} = 3$ $3^{2xy} = 3$ $2xy = 1$ From (2), $2x = \frac{1}{y}$ Put (3) into (1) $\frac{3}{y} = y-2$ $y^{2}-2y-3=0$ $(y-3)(y+1)=0$ $y=3$, $y=\ 1$	 Form a linear equation Form a non-linear equation Apply elimination method to solve simultaneous equations Obtain A and B

$x = \frac{1}{6}$, $x = -\frac{1}{2}$ Distance between 2 points $= \sqrt{\left(\frac{1}{6}+\frac{1}{2}\right)^2 + (3+1)^2}$ $= \sqrt{\frac{148}{9}}$ $= \frac{2\sqrt{37}}{3}$ units or 4.06 units	Apply distance formula correctly

7a. A toy car moved at a speed of $(2+\sqrt{3})$ cm per second from point M to point N. Given that the distance covered was $(2\sqrt{75}-1)$ cm, find the time taken to move from point M to point N in the form $a\sqrt{3}+b$, where a and b are constants. [4]

Time taken $= \frac{2\sqrt{75}-1}{(2+\sqrt{3})}$ $= \frac{(10\sqrt{3}-1)(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}$ $= \frac{20\sqrt{3}-30-2+\sqrt{3}}{4-3}$ $= 21\sqrt{3}-32$ seconds	Ratio of distance to speed Rationalise correctly Reduce denominator to 1

7b. Find the range of values of x that will satisfy the following inequalities, $2x+5>4$ and $6-2x^2 \geq 3+x$. [4]

$2x+5>4$ $2x>-1$ $x>-\frac{1}{2}$  $6-2x^2 \geq 3+x$ $2x^2+x-3 \leq 0$ $(2x+3)(x-1) \leq 0$ $-\frac{3}{2} \leq x \leq 1$ To satisfy both inequalities: $-\frac{1}{2} < x \leq 1$	Obtain $x >$ ½ Solve quadratic inequality

8. The quadratic equation $2x^2-2x-1=0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

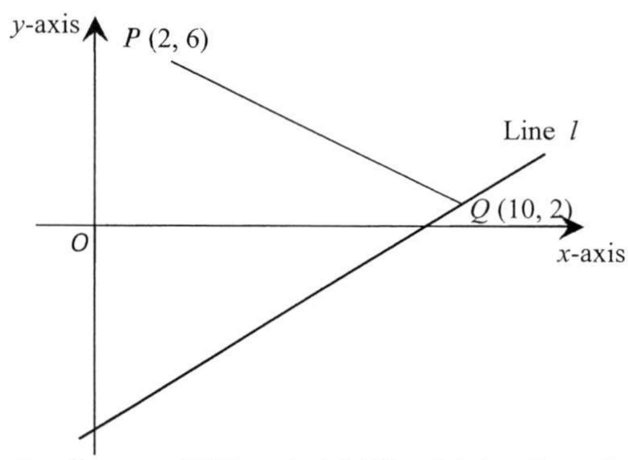
(i) Find the value of $\alpha^2+\beta^2$. [4]

Sum of roots $\frac{1}{\alpha}+\frac{1}{\beta}=1$ $\alpha+\beta=\alpha\beta$ Product of roots $\frac{1}{\alpha\beta}=-\frac{1}{2}$ $\alpha\beta=-2$ $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ $=(-2)^2-2(-2)$ $=8$	Equate sum of roots =1 Equate product of roots = $-\frac{1}{2}$

(ii) Find the quadratic equation in x whose roots are α^3 and β^3. [4]

Sum of roots $=\alpha^3+\beta^3$ $=(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)$ $=(-2)(8-[-2])$ $=-20$ Product of roots $=(\alpha\beta)^3$ $=(-2)^3$ $=-8$ Equation is $x^2+20x \quad 8=0$.	Equation '= 0'

9.



In the diagram, PQ is a straight line joining the points $P(2, 6)$ and $Q(10, 2)$.
Line l is parallel to the line $2y = x - 4$ and passes through point Q.
Given that the perpendicular bisector of PQ intersects Line l at point R,

(i) find the coordinates of point R, [6]

Midpoint of $PQ = (6, 4)$ Gradient of line $PQ = \frac{6-2}{2-(4)}$ $= -\frac{1}{2}$ Gradient of line perpendicular to $PQ = 2$ Equation of perpendicular bisector: $\frac{y-4}{x-6} = 2$ $y = 2x - 8$ Gradient of $RQ = \frac{1}{2}$ (given line is // to $2y = x - 4$) Equation of line RQ: $\frac{y-2}{x-10} = \frac{1}{2}$ $2y = x - 6$ Put (1) into (2) $2(2x\quad 8) = x\quad 6$ $x = \frac{10}{3}$ $y = 2\left(\frac{10}{3}\right) - 8$ $y = -\frac{4}{3}$ 4 $R = (\frac{10}{3}, -\frac{4}{3})$	Find midpt of PQ Apply gradient formula Obtain equation Obtain equation of RQ Solve simultaneous eq Obtain point R

(ii) calculate the area of the quadrilateral $PQRO$. [3]

	Area of ⊗ $TQM = \frac{1}{2}\begin{vmatrix} 2 & 0 & \frac{10}{3} & 10 & 2 \\ 6 & 0 & \frac{-4}{3} & 2 & 6 \end{vmatrix}$ $= \frac{1}{2}\left[\left(\frac{20}{3} + 60\right) - \left(-\frac{40}{3} + 4\right)\right]$ $= 38 \text{ units}^2$	Apply 'shoe-laced' in anti-clockwise direction to find area evaluate

10. The equation of a curve $y = ax^2 + 2x + 6$ can be written in the form $y = 3(x + b)^2 + c$, where a, b and c are constants.

(i) State the value of a .

Expressing $y = ax^2 + 2x + 6$ in the form $y = 3(x+b)^2 + c$, show that $b = \frac{1}{3}$ and find c . [5]

$a = 3$ $y = a\left[x^2 + \frac{2}{a}x + \frac{6}{a}\right]$	Complete the square
$y = a\left[\left(x + \frac{2}{2a}\right)^2 + \frac{6}{a} - \left(\frac{2}{2a}\right)^2\right]$ $y = a\left(x + \frac{1}{a}\right)^2 + 6 - \frac{1}{a}$	Obtain the square term
$b = \frac{1}{a}$, $b = \frac{1}{3}$ (shown)	Able to show $b = \frac{1}{3}$
$c = 6 - \frac{1}{(3)}$, $c = \frac{17}{3}$	Obtain c

Hence,

(ii) find the greatest value of $\frac{2}{y}$. Explain your choice for the value of y . [3]

For y to be least, $3(x+b)^2 = 0$, $4y = c$	Explain the choice of y
Greatest value of $\frac{2}{y} = \frac{2}{(\frac{17}{3})}$ $= \frac{6}{17}$	Substitute correct y value

(iii) determine with explanation the number of points of intersection between the curve and the x-axis. [2]

$3x^2 + 2x + 6 = 0$	
Discriminant $= 2^2 \quad 4(3)(6)$ $= \quad 68$	Find discriminant
Since discriminant < 0, then there are no real roots. Thus there is no intersection point between the curve and the x-	Show discriminant < 0 and explain

axis. **<u>Or</u>** Put $y = 0$ (intersection with x-axis) $3x^2 + 2x + 6 = 0$ $x = \dfrac{-2 \pm \sqrt{4 - 4(3)(6)}}{2(3)}$ $x = \dfrac{-2 \pm \sqrt{-68}}{2(3)}$ No solution, then there is no intersection point between the curve and the x-axis.	solve the quadratic equation to obtain no solution explanation

11. The polynomial $g(x) = x^3 + ax^2 - bx - 2$ has a factor $(x + 1)$ and it leaves a remainder of 24 when divided by $(x \quad 2)$.

(i) Show that $a = 4$ and $b = \quad 1$. [4]

$(x + 1)$ is a factor, by Factor Thm, $g(\; 1) = 0$	Apply Factor Thm

$1 + a + b \quad 2 = 0$ $a + b = 3$ (1) By Remainder Thm, $g(2) = 24$ $2^3 + 4a \quad 2b \quad 2 = 24$ $2a \quad b = 9$ (2) $(1) + (2) \quad 3a = 12$ $a = 4$ (shown) $4 + b = 3, \quad b = \quad 1$ (shown)	Obtain eqn (1) Apply Remainder Thm Obtain eqn (2)

(ii) Taking $a = 4$ and $b = \quad 1$, solve the equation $g(x) = 0$, leaving your answers in the exact form.
Hence, find the integer value of x for which $(x-2)^3 + 4(x-2)^2 + x - 4 = 0$. [7]

$g(x) = x^3 + 4x^2 + x - 2$ $x^3 + 4x^2 + x - 2 = (x+1)(x^2 + px - 2)$ where p is a constant compare coefficient of x: $-2 + p = 1$ $p = 3$ $\therefore g(x) = (x+1)(x^2 + 3x - 2)$ $g(x) = 0, \ x = -1, \quad x = \frac{-3 \pm \sqrt{9+8}}{2}$ $x = \frac{-3 \pm \sqrt{17}}{2}$ $(x-2)^3 + 4(x-2)^2 + x - 4 = 0$ $(x-2)^3 + 4(x-2)^2 + ((x-2) - 2 = 0$ Put $x = x - 2$ when $x = -1, \quad x - 2 = -1$ The integer value of $x = 1$	Obtain correct coefficient of x^2 and constant, $\quad 2$. $x = \quad 1$ apply quadratic formula to solve 2 other roots 2 correct answers Show the 2 equations are linked

12a. Solve $5^{x+1} - 2(5^{-x}) = 9$. [6]

$5^{x+1}-2(5^{-x})=9$ $5(5^x)-\frac{2}{5^x}=9$ Let $n=5^x$ $5n-\frac{2}{n}=9$ $5n^2-9n-2=0$ $(5n+1)(n-2)=0$ $n=2$, $n=-\frac{1}{5}$ $\therefore 5^x=2$ $5^x=-\frac{1}{5}$(rejected) $x\lg 5=\lg 2$ OR $x\ln 5=\ln 2$ $x=\frac{\lg 2}{\lg 5}$ $x=0.431$	Apply Indices Law to split the terms correctly Use substitution to form a quadratic equation Obtain $5^x=2$ & $5^x=$ 0.2 Solve by bringing both sides to lg or ln

12b. Express $\frac{4x^3+2x-1}{(2x-1)(x+1)^2}$ in partial fractions. [6]

Let Let $\frac{4x^3+2x-1}{(2x-1)(x+1)^2}=2+\frac{A}{2x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$, where A, B and C are constants	Obtain '2' either by inspection or long division Obtain 3 partial fractions
$4x^3+2x-1=2(2x-1)(x+1)^2+A(x+1)^2+B(2x-1)(x+1)+C(2x-1)$ $x=-1,$ $-4-2-1=-3C$ $\therefore C=\frac{7}{3}$ $x=\frac{1}{2},$ $\frac{1}{2}+1-1=\frac{9}{4}A$ $\therefore A=\frac{2}{9}$ $x=0,$ $-1=-2+A-B-C$ $\therefore B=-\frac{28}{9}$ $\frac{4x^3+2x-1}{(2x-1)(x+1)^2}=2+\frac{2}{9(2x-1)}-\frac{28}{9(x+1)}+\frac{7}{3(x+1)^2}$	Obtain correct values of A, B & C Express as partial fractions